Chapter 8

Electromagnetic Waves

Maxwell's field equations provide us with all the information that can be derived from the classical theory of electric and magnetic fields. What in particular is of paramount importance is that the fields generated by the moving charges can leave the source to propagate through space in the form of waves, called *electromagnetic waves*. The situation finally culminated in the identification of light waves with electromagnetic waves and the consequent development of **electromagnetic theory** of light.

8.1 Pre-Maxwell Electrodynamics

Before the appearance of Maxwell in the scene, the basic laws of electricity and magnetism were expressed by the following equations.

1. $\vec{\nabla}\cdot\vec{D}=\rho \quad \Rightarrow \vec{\nabla}\cdot\vec{E}=\rho/\varepsilon$ (8.1.1)

2. $\vec{\nabla}\cdot\vec{B}=0 \quad \Rightarrow \vec{\nabla}\cdot(\mu\vec{H})=0 \Rightarrow \vec{\nabla}\cdot\vec{H}=0$ (8.1.2)

3. $\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t \Rightarrow \vec{\nabla}\times\vec{E}=-\mu(\partial\vec{H}/\partial t)$ (8.1.3)

4. $\vec{\nabla}\times\vec{H}=\vec{J}$ (8.1.4)

where $\vec{D}$ is the electric displacement vector $=\varepsilon\vec{E}$, ε being the permittivity of the medium; ρ the charge density; $\vec{B}$ the magnetic induction vector $=\mu\vec{H}$, μ being the permeability of the medium; $\vec{E}$ the electric field vector; $\vec{H}$ the magnetic field vector and $\vec{J}$ the conduction current density $=\sigma\vec{E}$, σ being the conductivity of the medium.

- Of the above four equations, the first three are general equations valid for both static and dynamic fields. The fourth equation was derived from steady state observations—the *Ampere's law.*
- A critical examination shows that the ***Ampere's law is bound to fail in time-varying fields.*** Taking the divergence of both sides of (8.1.4), we get

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=\vec{\nabla}\cdot\vec{J}=0 \tag{8.1.5}$$

since the divergence of a curl always vanishes.

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This is true for steady-state phenomenon. It is however incompatible, when current changes with time, with the *equation of continuity* that reflects the principle of conservation of charge :

$$\boxed{\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0.} \tag{8.1.6}$$

8.1.1 Maxwell's contribution

Maxwell could understand the situation well and suggested that the way out of the impasse lies in modifying Ampere's law which suffers from an incomplete definition of the total current density.

By using Gauss' law, the relation (8.1.6) may be written as

$$\vec{\nabla}\cdot\vec{J}=-\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})=\vec{\nabla}\cdot\left(-\frac{\partial\vec{D}}{\partial t}\right)$$

$$\Rightarrow\quad \vec{\nabla}\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)=0 \tag{8.1.7}$$

Maxwell, therefore, replaced $\vec{J}$ of Ampere's law by $\vec{J}+\dfrac{\partial\vec{D}}{\partial t}$ i.e., $\vec{J}+\varepsilon\dfrac{\partial\vec{E}}{\partial t}$ and modified the Ampere's law as :

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}=\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t} \tag{8.1.8}$$

which then becomes valid for steady-state phenomena and is also compatible with the equation of continuity.

The *modified* Ampere's law has now *two* terms—the term $\vec{J}$ is the usual '*conduction current density*'; the second term $\partial\vec{D}/\partial t$, arising from time-variation of electric displacement, is called '*displacement current density*' by Maxwell. The introduction of this '*displacement current*' *is the chief contribution of* Maxwell to *electromagnetic theory.*

Displacement current—But what exactly is meant by *displacement current?* In fact, displacement current, is *not* current in the sense of the motion of charges.

Consider the simple circuit as illustrated in Fig.8.1 consisting of a charged capacitor C with its plates connected by a conducting wire. The current that flows in the wire is equal to the time rate of change of charge on the plates, i.e.,

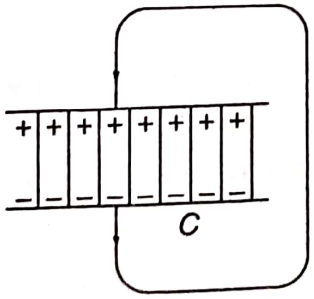

Fig 8.1

$$I=\frac{dQ}{dt} \tag{8.1.9}$$

where Q is the charge on the positive plate of the capacitor and is related to the field E in the capacitor by

$$E=\frac{\sigma}{\varepsilon_0}=\frac{Q}{\varepsilon_0 A} \tag{8.1.10}$$

where σ is the charge density and A is the area of the plate.

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$$\therefore \quad I = \frac{d}{dt}(Q) = \frac{d}{dt}(\varepsilon_0 AE), \text{ using (8.1.10)}$$

$$= \varepsilon_0 A \frac{\partial E}{\partial t} = A\frac{\partial D}{\partial t}$$

$$\Rightarrow \quad \frac{\partial D}{\partial t} = \frac{I}{A}, \text{ the } \textit{current density}. \qquad (8.1.11)$$

The quantity $\partial D/\partial t$ can thus be *interpreted* as the *density of some current*. This corresponds to the "current" that must flow in the space, including vacuum, between the charged plates of the capacitor when joined by a conducting wire, thus completing the conduction current.

8.1.2 Dimensions of displacement current density

To find the dimensions of displacement current density $\partial\vec{D}/\partial t$, we start as

$$\vec{D} = \varepsilon\vec{E} = \text{permittivity} \times \text{electric field}$$
$$= (\text{Farad/metre}) \times (\text{volt/metre})$$
$$= \text{Coulomb}/(\text{metre})^2$$

$$\therefore \text{ Dimensions of } \frac{\partial\vec{D}}{\partial t} = \frac{1}{\text{sec.}} \cdot \frac{\text{Coulomb}}{(\text{metre})^2} = \frac{\textbf{Ampere}}{\textbf{(metre)}^{\textbf{2}}},$$

the same as of current per unit area or the current density.

Relative size of two types of currents—Consider a metallic wire in which there is an electric field given by

$$\vec{E} = \vec{E}_0 e^{-j\omega t}.$$

$$\therefore \quad \vec{J} = \sigma\vec{E} = \sigma\vec{E}_0 e^{-j\omega t} \Rightarrow |\vec{J}|^2 = \sigma^2|\vec{E}_0|^2$$

$$\vec{D} = \varepsilon_0\vec{E}_0 e^{-j\omega t} \Rightarrow \frac{\partial\vec{D}}{\partial t} = -j\omega\varepsilon_0\vec{E}_0 e^{-j\omega t}$$

$$\Rightarrow \left|\frac{\partial\vec{D}}{\partial t}\right|^2 = \omega^2\varepsilon_0^2|\vec{E}_0|^2 \qquad (8.1.12)$$

$$\therefore \quad \frac{|\vec{J}|}{|\partial\vec{D}/\partial t|} = \frac{\sigma}{\omega\varepsilon_0}$$

If, for instance, the metal be copper, $\sigma = 5.9 \times 10^7$, then

$$\frac{\sigma}{\omega\varepsilon_0} \sim \frac{10^{19}}{\omega}$$

This shows that the ratio is very large for all frequencies and it implies that $|\vec{J}| \gg |\partial\vec{D}/\partial t|$. So, the displacement current is not as significant as the conduction current. Nevertheless, the very idea and introduction of displacement current has been of far-reaching consequences in the development of physics and a rich variety of new phenomena was unravelled by the use of displacement current.

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8.2 Maxwell's field equations : Physical significance

The four equations which the field vectors $\vec{E}, \vec{D}, \vec{B}$ and $\vec{H}$ satisfy everywhere are the following.

1. $\vec{\nabla}\cdot\vec{D} = \rho$ or, $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon$. **(Gauss' law of electrostatics)**
2. $\vec{\nabla}\cdot\vec{B} = 0$ or, $\vec{\nabla}\cdot\vec{H} = 0$ **(Gauss' law of magnetostatics)**
3. $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ or, $\vec{\nabla}\times\vec{E} = -\mu\dfrac{\partial\vec{H}}{\partial t}$ **(Faraday's law of induction)**
4. $\vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t}$ or, $\vec{\nabla}\times\vec{H} = \vec{J} + \varepsilon\dfrac{\partial\vec{E}}{\partial t}$ **(Modified Ampere's law)**

These equations constitute the fundamental equations of electromagnetic field and are known as **Maxwell's field equations**. Of these, note that the first two are *steady state equations* and the last two are *time-varying equations.*

Physical significance—The equation 1 is a restatement of *Gauss' law in electrostatics,* in the differential form.

The equation 2 is a restatement of *Gauss' law in magnetism, along with the fact* that magnetic poles always exist in pairs, i.e., *monopole does not exist.*

The equation 3, the first time-varying equation, is a restatement of *Faraday's law of electromagnetic induction.*

And the equation 4, the second time-varying equation, is a restatement of *Ampere's law modified by the introduction of displacement current* along with the conduction current. It tells us that *a changing electric field produces a changing magnetic field.* just as a changing magnetic field produces a changing electric field. And *an alternate production of electric and magnetic field* gives rise to the propagation of *electromagnetic waves.*

- Integrating equation 1, over a volume V, we have

$$\frac{1}{\varepsilon}\int_V \rho dV = \int_V \vec{\nabla}\cdot\vec{E}\,dV = \oint_S \vec{E}\cdot d\vec{S}, \text{ by divergence theorem.}$$

But, $\oint_S \vec{E}\cdot d\vec{S}$ is the electric flux through a closed surface and $\int_V \rho dV = q$, the total charge within the closed surface. So, $\oint_S \vec{E}\cdot d\vec{S} = q/\varepsilon$, which is the *Gauss' law.*

- Integrating equation 2, over a volume V,

$$0 = \int_V \vec{\nabla}\cdot\vec{B}dV = \oint_S \vec{B}\cdot d\vec{S}, \text{ by divergence theorem.}$$

But, $\oint_S \vec{B}\cdot d\vec{S}$ is the magnetic flux through a closed surface and is zero, implying that the flux entering into the volume equals the flux leaving it. So, there is no *source* or *sink* in the volume i.e., *monopole cannot exist.*

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- Integrating equation 3, over any closed surface bounding a path C

$$-\oint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} = \oint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = \oint_C \vec{E}\cdot d\vec{l}, \text{ by Stokes' theorem.}$$

But $\oint_C \vec{E}\cdot d\vec{l}$ is the induced emf in a closed circuit and $\oint_S -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$ is the *negative* rate of change of magnetic flux through the circuit and they are equal. This is *Faraday's law* of electromagnetic induction.

- The basic laws of electricity and magnetism are thus contained succinctly in the four equations of Maxwell, along with an important of Ampere's law.

8.3 Component forms of Maxwell's equations

The Maxwell's four equations may also be given the component forms in terms of the Cartesian coordinates and are shown below.

1. $$\vec{\nabla}\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}E_x+\hat{j}E_y+\hat{k}E_z\right)$$

$$= \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}$$

$$\therefore \boxed{\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_x}{\partial z}=\frac{\rho}{\varepsilon}} \quad (8.3.1)$$

2. $$\vec{\nabla}\cdot\vec{H} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}H_x+\hat{j}H_y+\hat{k}H_z\right)$$

$$= \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}$$

$$\therefore \boxed{\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0} \quad (8.3.2)$$

3. $$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\hat{j}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{k}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

Also, $$-\mu\frac{\partial \vec{H}}{\partial t} = -\mu\left(\hat{i}\frac{\partial H_x}{\partial t}+\hat{j}\frac{\partial H_y}{\partial t}+\hat{k}\frac{\partial H_z}{\partial t}\right)$$

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Equating the components along the corresponding co-ordinates, we get

$$\left.\begin{aligned} -\mu\frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ -\mu\frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ -\mu\frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{aligned}\right\} \quad (8.3.3)$$

4. $$\vec{\nabla}\times\vec{H} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{j}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{k}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

Also, $$\varepsilon\frac{\partial\vec{E}}{\partial t} = \varepsilon\left(\hat{i}\frac{\partial E_x}{\partial t} + \hat{j}\frac{\partial E_y}{\partial t} + \hat{k}\frac{\partial E_z}{\partial t}\right)$$

For a non-conducting medium, however, we have

$$\vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} \quad (\because\ J = 0)$$

$\therefore$ Equating the components along the corresponding co-ordinate axes,

$$\left.\begin{aligned} \varepsilon\frac{\partial E_x}{\partial t} &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \varepsilon\frac{\partial E_y}{\partial t} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \varepsilon\frac{\partial E_z}{\partial t} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{aligned}\right\} \quad (8.3.4)$$

8.4 Decay of free charge : Relaxation time

An important deduction that follows immediately from Maxwell's equation is the decay of free charges.

From the fourth equation of Maxwell, we have

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}$$

$\therefore\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \sigma(\vec{\nabla}\cdot\vec{E}) + \varepsilon\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})$, assuming σ and ε as constants.

or, $$0 = \sigma\frac{\rho}{\varepsilon} + \frac{\partial\rho}{\partial t} \quad (\because\ \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon)$$

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Integrating, we obtain

$$\boxed{\rho = \rho_0 e^{-t\sigma/\varepsilon} = \rho_0 e^{-t/\tau}} \tag{8.4.1}$$

where $\tau = \varepsilon/\sigma$ and is called the **relaxation time**.

Eq. (8.4.1) shows that *free charge decays exponentially* at a rate independent of any other electromagnetic disturbance.

8.5 Wave equation in a dielectric (non-conducting) medium

We shall now see that as a result of Maxwell's set of equations, the electric and magnetic fields can travel through space in the form of waves, called *electromagnetic waves* (e.m. waves). We shall derive hereunder the relevant wave equation in a dielectric (i.e., non-conducting) medium.

Let μ be the permeability, σ the conductivity, ε the permittivity of a *dielectric* medium, assumed linear, i.e., ε, μ and σ are constants. As it is non-conducting, the conductivity $\sigma = 0$. Let it also be free from any charge source implying that charge density $\rho = 0$. Thus, in such a medium, Maxwell's equations for the electric field vector $\vec{E}$ and magnetic field vector $\vec{H}$ are :

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0 \\ \vec{\nabla}\cdot\vec{H} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\mu\frac{\partial\vec{H}}{\partial t} \\ \vec{\nabla}\times\vec{H} &= \varepsilon\frac{\partial\vec{E}}{\partial t} \end{aligned}} \qquad \begin{matrix} (8.5.1) \\ (8.5.2) \\ (8.5.3) \\ (8.5.4) \end{matrix}$$

since in a dielectric medium the current density $\vec{J} = \sigma\vec{E} = 0$.

Wave equation for $\vec{E}$: From (8.5.3), taking curl, we have,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) = -\mu\left(\vec{\nabla}\times\frac{\partial\vec{H}}{\partial t}\right) = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

$$\text{or,}\quad \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{E}(\vec{\nabla}\cdot\vec{\nabla}) = -\mu\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\vec{E}}{\partial t}\right), \quad \text{using (8.5.4)}$$

$$\text{or,}\quad -\nabla^2\vec{E} = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad (\because\ \vec{\nabla}\cdot\vec{E} = 0 \quad \text{by (8.5.1)})$$

$$\therefore\quad \boxed{\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2 E}{\partial t^2}} \tag{8.5.5}$$

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Wave equation for $\vec{H}$: From (8.5.4), taking curl, we have,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times \left(\varepsilon \frac{\partial \vec{E}}{\partial t}\right) = \varepsilon \left(\vec{\nabla} \times \frac{\partial \vec{E}}{\partial t}\right) = \varepsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

$$\text{or,} \quad \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \vec{H}(\vec{\nabla} \cdot \vec{\nabla}) = \varepsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \vec{H}}{\partial t}\right), \quad \text{using (8.5.3)}$$

$$\text{or,} \quad -\nabla^2 \vec{H} = -\mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}, \quad (\because \vec{\nabla} \cdot \vec{H} = 0 \quad \text{by (8.5.2)})$$

$$\therefore \quad \boxed{\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 H}{\partial t^2}} \tag{8.5.6}$$

Eq. (8.5.5) and (8.5.6) are respectively the *general wave equations* for the electric field $\vec{E}$ and the magnetic field $\vec{H}$ in a dielectric medium.

• Note that $\vec{E}$ and $\vec{H}$ satisfy the same wave equation and we can readily identify $v = \sqrt{1/\mu\varepsilon}$ as the speed of the wave.

8.6 Wave equation in a conducting medium

Let μ be the permeability, ε the permittivity and σ the conductivity of the medium which is free from any charge source so that the charge density $\rho = 0$.

For such a medium the Maxwell's equations take the form :

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{8.6.1}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{8.6.2}$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{8.6.3}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial L}{\partial t} \tag{8.6.4}$$

since the current density $\vec{J} = \sigma \vec{E}$, $\vec{E}$ and $\vec{H}$ being the electric and magnetic field vectors.

Wave equation for $\vec{E}$: From (8.6.3), taking curl, we obtain

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu \frac{\partial \vec{H}}{\partial t}\right) = -\mu \left(\vec{\nabla} \times \frac{\partial \vec{H}}{\partial t}\right) = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$\text{or,} \quad \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{E}(\vec{\nabla} \cdot \vec{\nabla}) = -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right), \quad \text{using (8.6.4).}$$

$$\text{or,} \quad -\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}, \quad \text{using (8.6.1).}$$

$$\therefore \quad \boxed{\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}} \tag{8.6.5}$$

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Wave equation for $\vec{H}$: From (8.6.4), taking curl we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times \left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) = \sigma (\vec{\nabla} \times \vec{E}) + \varepsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

or, $\vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \vec{H}(\vec{\nabla} \cdot \vec{\nabla}) = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$, using (8.6.3).

or, $-\nabla^2 \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2}$ $(\because \vec{\nabla} \cdot \vec{H} = 0,$ by (8.6.2))

$$\therefore \quad \boxed{\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}} \tag{8.6.6}$$

- Note that here also $\vec{E}$ and $\vec{H}$ satisfy the same wave equation.
- If the medium is *non-conducting*, i.e., a *dielectric*, $\sigma = 0$ and (8.6.5) and (8.6.6) reduce respectively to the following form

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{8.6.7}$$

$$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \tag{8.6.8}$$

in agreement with what we obtained earlier in (8.5.5) and (8.5.6).

Dissipative term—Comparing (8.6.5) and (8.6.7), the term $\mu\sigma(\partial \vec{E}/\partial t)$ is the *dissipative term* that allows current to flow. Similarly, by comparing (8.6.6) and (8.6.8), we see that the *dissipative term* is $\mu\sigma(\partial \vec{H}/\partial t)$.

8.7 Velocity (or speed) of electromagnetic waves

We have for a *dielectric medium*, the following wave equations :

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{8.7.1}$$

$$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \tag{8.7.2}$$

for the electric field $\vec{E}$ and the magnetic field $\vec{H}$ respectively.

They are of the form of the familiar wave equation :

$$\nabla^2 \phi = \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} \tag{8.7.3}$$

where v is the velocity (or speed) of propagation of the wave.

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Comparing (8.7.1) and (8.7.2) with (8.7.3), it follows that *any time variation in electric and magnetic fields propagates with the same velocity*

$$\boxed{v = \frac{1}{\sqrt{\mu\varepsilon}}} \tag{8.7.4}$$

• One should bear in mind that while equation (8.7.3) is a *scalar* equation, each of (8.7.1) and (8.7.2) is a *vector* equation. The implication is straight-forward. Each component of $\vec{E}$ and $\vec{H}$ satisfies the scalar equation (8.7.3).

Speed of e.m. waves in vacuo — According to Maxwell's equations, therefore, the empty space supports the propagation of electromagnetic waves with a speed

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.8547 \times 10^{-12}}}$$
$$= \mathbf{2.99784 \times 10^8\ m/s.}$$

where $\mu_0 = 4\pi \times 10^{-7}$ and $\varepsilon_0 = 8.8547 \times 10^{-12}$ in SI units.

This is precisely the *speed of light* c in vacuum! Maxwell's equations thus vindicate that *light is an electromagnetic wave.* The *role played* by Maxwell's term $\mu\varepsilon(\partial\vec{E}/\partial t)$, in modified Ampere's law, in this regard is *crucial.* For without it, the wave equation would *not* have emerged at all.

Refractive index—We have seen that the electromagnetic waves propagate through a homogeneous (linear) medium with a speed v given by

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{8.7.5}$$

If the *medium is transparent*, μ is very nearly 1 (i.e., $\mu \simeq 1$) and ε greater than 1 ($\varepsilon > 1$). This implies that light should travel slower through matter, as is well-known from optics :

$$v = \frac{c}{n} \tag{8.7.6}$$

where c is the velocity of light in vacuo and n the refractive index of the material. So from (8.7.5) and (8.7.6) the index n is related to electric and magnetic properties of the material

$$\therefore\ n = \frac{c}{v} = \frac{1/\sqrt{\mu_0\varepsilon_0}}{1/\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \tag{8.7.7}$$

since in a non-magnetic dielectric, $\mu = \mu_0$.

$$\therefore\ \boxed{n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{K}} \tag{8.7.8}$$

where the relative permittivity or $\varepsilon/\varepsilon_0 = K$, the dielectric constant of the medium.

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• We know, again, that n varies with the wavelength λ or the *colour* of the incident light and this fact accounts for the **dispersion** of light. The electromagnetic explanation for the dispersion is also forthcoming, since the electric polarizability of a molecule, and also therefore the permittivity ε, depends on the rapidity of the oscillation of electric field vector $\vec{E}$.

• Eq. (8.7.8), i.e., $n^2 = K$ has been verified experimentally in case of non-polar molecules such as air, H_2, C_6H_6 etc.

8.8 Solution of wave equation : Dielectric medium

Consider the wave equation in a homogeneous, isotropic, linear dielectric medium.

The simplest type of wave which would be a solution of (8.7.1) or (8.7.2) is a **plane wave** — a good approximation to actual waves in many situations, and many elementary but important characteristics of e.m. waves can be obtained from it. We shall therefore consider plane wave solutions of (8.7.1) and (8.7.2).

In a **plane wave**, the wave amplitudes — the field vector components — are constants at a given time over all points in a plane perpendicular to the direction of propagation. This plane is called the *wavefront* that advances with a velocity v (Fig. 8.2) in a direction normal to it.

Plane wave solution — For the propagation of wave in any direction, we take the plane wave fields to be of the form

$$\vec{E} = \vec{E}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)} \qquad (8.8.1)$$

$$\vec{H} = \vec{H}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)} \qquad (8.8.2)$$

where $\vec{E}_0$ and $\vec{H}_0$ ae complex amplitudes, fixed in space and time and $\vec{k}$ is the propagation vector determining the direction of wave propagation.

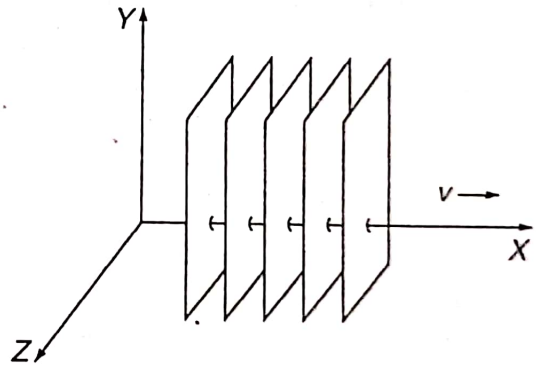

Fig 8.2

Substituting (8.8.1) in Maxwell's equations, $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{H} = 0$, we obtain

$$\vec{k}\cdot\vec{E} = 0 \qquad (8.8.3)$$

$$\text{and} \quad \vec{k}\cdot\vec{H} = 0 \qquad (8.8.4)$$

It thus follows that both $\vec{E}$ and $\vec{H}$ are perpendicular to the direction of propagation vector $\vec{k}$. So, **an electromagnetic wave is a transverse wave.**

On substitution of (8.8.1) in Maxwell's equations

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\text{and} \quad \vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t}$$

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$$j\vec{k} \times \vec{E} = -\mu(-j\omega\vec{H}) \Rightarrow \vec{k} \times \vec{E} = \mu\omega\vec{H} \quad (8.8.5)$$

$$j\vec{k} \times \vec{H} = \varepsilon(-j\omega\vec{E}) \Rightarrow \vec{k} \times \vec{H} = -\varepsilon\omega\vec{E} \quad (8.8.6)$$

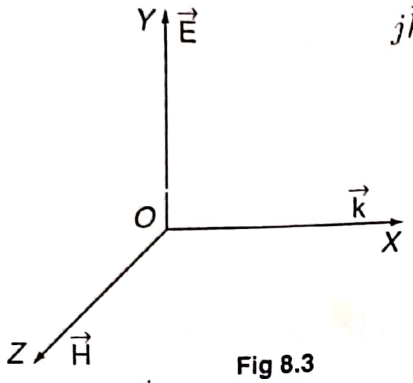

Fig 8.3

Eq. (8.8.5) shows that $\vec{H}$ is normal to both $\vec{k}$ and $\vec{E}$, while (8.8.6) implies that $\vec{E}$ is normal to both $\vec{k}$ and $\vec{H}$. It therefore follows that *the field vectors* $\vec{E}$ *and* $\vec{H}$ *are mutually perpendicular and they are also perpendicular to the direction of propagation* $\vec{k}$. *So* $\vec{E}, \vec{H}$ *and* $\vec{k}$ *consttute a right handed orthogonal system* (Fig. 8.3).

• In taking the plane wave solutions (8.8.1) and (8.8.2) we assumed $\vec{E}$ and $\vec{H}$ to be in phase. *But are they really in phase?* The answer follows.

If we substitute (8.8.1) in the wave equations (8.5.5) and (8.5.6)

$$k^2 = \varepsilon\mu\omega^2 \quad (8.8.7)$$

Thus, for e.m. waves in a linear dielectric medium, k is real and hence (8.8.5) implies that $\vec{E}$ *and* $\vec{H}$ *are in phase.*

Since $\vec{E}$ is normal to $\vec{k}$, we have from (8.8.7), in terms of magnitudes,

$$\begin{aligned} & kE = \mu\omega H \\ \text{or,} \quad & \sqrt{\varepsilon}E = \sqrt{\mu}H \\ \Rightarrow \quad & \tfrac{1}{2}\varepsilon E^2 = \tfrac{1}{2}\mu H^2 \end{aligned} \quad (8.8.8)$$

So, in a linear dielectric medium, the electromagnetic energy is equally shared between the electric and magnetic fields.

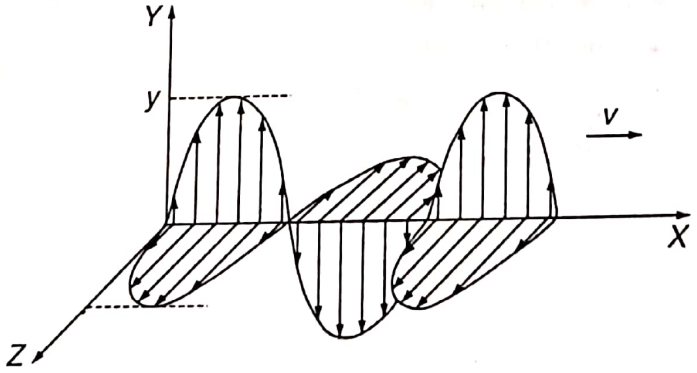

Fig 8.4

Fig. 8.4 shows the diagram of a TEM (transverse electro-magnetic) wave propagating along $+x$ direction at any time t. $\vec{E}$ at different x-values are indicated by arrowheads, parallel to y-axis, in xy-plane. $\vec{H}$ are also shown at those x-values by arrowheads, but parallel to z-axis, and lyng in zx-plane.

• The above analysis is also applicable to free space. Only ε and μ are to be replaced by ε_0 and μ_0 rrespectively.

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• The ratio of the amplitudes of $\vec{E}$ and $\vec{H}$ at any point and at any instant, is

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu}{\varepsilon}} = Z_0, \text{ say} \tag{8.8.9}$$

using the equation (8.8.8).

Z_0 has the dimensions of impedence and is called the *intrinsic impedance* of the medium concerned.

In free space, $Z_0 = \sqrt{\mu_0/\varepsilon_0} = \mathbf{377\,\Omega}$, putting the values of μ_0 and ε_0. We shall return to Z once again later.

8.9 Poynting vector and Poynting theorem

Poynting vector—Poynting vector is *defined as the cross-product of the electric field vector $\vec{E}$ and the magnetic field vector $\vec{H}$* and is symbolised by $\vec{S}$.

$$\therefore \quad \textbf{Poynting vector,} \quad \boxed{\vec{S} = \vec{E} \times \vec{H}}$$

For an electromagnetic wave, the direction of propagation of the wave is always at right angles to both $\vec{E}$ and $\vec{H}$; so the direction of Poynting vector is *always in the direction of propagation of the wave.*

But *what does this Poynting vector signify or represent*? It measures the electric power (or electric energy per unit time) per unit area for,

$$\vec{E} \times \vec{H} = \frac{\text{volt}}{\text{length}} \times \frac{\text{current}}{\text{length}} = \frac{\text{power}}{\text{area}}$$

The SI unit of Poynting vector is Watt/sq. metre ($\mathrm{Wm^{-2}}$).

Thus, the **significance** of the *Poynting vector* is that it *measures the flow of electromagnetic energy per unit time per unit area placed normally to the direction of propagation of the electromagnetic wave.*

Poynting theorem—As the electromagnetic wave propagates from one point to another, there occurs a transit of electromagnetic energy. It is observed that in the process, the *law of conservation of energy* holds good and is referred to as **Poynting's theorem.**

From Maxwell's field equations, we have

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}; \quad \vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{8.9.1}$$

$$\therefore \quad \vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial \vec{H}}{\partial t} \cdot \vec{H} = -\tfrac{1}{2}\mu \frac{\partial H^2}{\partial t} \tag{8.9.2}$$

$$\text{and} \quad \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t} = \vec{E} \cdot \vec{J} + \tfrac{1}{2}\varepsilon \frac{\partial E^2}{\partial t} \tag{8.9.3}$$

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Subtracting (8.9.2) from (8.9.3), we obtain

$$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) - \vec{H}\cdot(\vec{\nabla}\times\vec{E}) = \vec{E}\cdot\vec{J} + \tfrac{1}{2}\varepsilon\frac{\partial E^2}{\partial t} + \tfrac{1}{2}\mu\frac{\partial H^2}{\partial t}$$

$$= \vec{E}\cdot\vec{J} + \frac{1}{2}\frac{\partial}{\partial t}(\varepsilon E^2 + \mu H^2) \qquad (8.9.$$

Using on L.H.S. of (8.9.4) the vector identity

$$\vec{A}\cdot(\vec{B}\times\vec{C}) - \vec{C}\cdot(\vec{B}\times\vec{A}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = -\vec{B}\cdot(\vec{A}\times\vec{C}),$$

we obain :

$$-\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{E}\cdot\vec{J} + \frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2\right)$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{S} + \frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2\right) = -\vec{E}\cdot\vec{J} \qquad (8.9.$$

Integrating (8.9.5) over a volume V bounded by a closed surface A,

$$\int_V \vec{\nabla}\cdot\vec{S}\,dV + \frac{\partial}{\partial t}\int_V \left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2\right) dV = -\int_V \vec{E}\cdot\vec{J}\,dV$$

$$\text{or,} \quad \int_A \vec{S}\cdot d\vec{A} + \frac{\partial}{\partial t}\int_V \left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2\right) dV = -\int_V \vec{E}\cdot\vec{J}\,dV \qquad (8.9.$$

applying Gauss's divergence theorem, $d\vec{A}$ being an elemental area of the surface A.

Eq. (8.9.6) is the *statement of the well-known* **Poynting's theorem.**

Let us now explore what (8.9.6) does **signify.**

The **first term** $\int_A \vec{S}\cdot d\vec{A}$ is the *amount of energy flowing out per unit time*, i.e rate of energy-flow out of the bounding surface A.

The **second term** $\frac{\partial}{\partial t}\int_V \left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2\right) dV$ is the *time-rate of change of the tot energy* (electric + magnetic) of the electromagnetic field stored in the volume V.

The **third term** $-\int_V \vec{E}\cdot\vec{J}\,dV$ is the *work done by the field* on the source with th energy dissipated in the form of *Joule heat* in the volume V.

Eq. (8.9.6) thus gives none else than the **law of conservation of energy.**

- In free-space, $\vec{J} = 0$ and (8.9.6) then reduces to

$$\int_A \vec{S}\cdot d\vec{A} = -\frac{\partial}{\partial t}\int_V \left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2\right) dV$$

which states that *the flow of energy per unit time across the boundary of a clos surface is equal to the time-rate of change of total energy of the electromagnetic fie in the volume enclosed by the surface.*

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8.10 Average values of Poynting vector

1. For sinusoidally time-varying fields—Let the electric and magnetic field vectors be represented as

$$\vec{E} = \vec{E}_0 \sin \omega t \; ; \quad \vec{H} = \vec{H}_0 \sin \omega t$$

The **average Poynting vector** is *defined as its average value over one complete period T of the electro-magnetic wave.*

$$\begin{aligned}
\therefore \quad \langle \vec{S} \rangle &= \frac{1}{T} \int_0^T (\vec{E} \times \vec{H}) dt \\
&= \frac{1}{T} (\vec{E}_0 \times \vec{H}_0) \int_0^T \sin^2 \omega t \, dt \\
&= \frac{\omega}{2\pi} (\vec{E}_0 \times \vec{H}_0) \int_0^{2\pi/\omega} \frac{1 - \cos 2\omega t}{2} dt \\
&= \frac{\omega}{4\pi} (\vec{E}_0 \times \vec{H}_0) \left[t - \frac{\sin 2\omega t}{2\omega} \right]_0^{2\pi/\omega} \\
&= \frac{\omega}{4\pi} (\vec{E}_0 \times \vec{H}_0) \frac{2\pi}{\omega} = \tfrac{1}{2} (\vec{E}_0 \times \vec{H}_0) = \frac{\vec{E}_0}{\sqrt{2}} \times \frac{\vec{H}_0}{\sqrt{2}} \\
&= \vec{E}_{\text{rms}} \times \vec{H}_{\text{rms}}
\end{aligned} \tag{8.10.1}$$

2. Time-varying fields : varying as $e^{-j\omega t}$ — Let the physical fields be given by the real parts of their complex representation, i.e.,

$$\begin{aligned}
&\text{Re}\, \vec{E} = \text{Re}\, \vec{E}_0 e^{-j\omega t}; \quad \text{Re}\, \vec{H} = \text{Re}\, \vec{H}_0 e^{-j\omega t} \\
\therefore \quad &\text{Re}\, \vec{E} = \tfrac{1}{2} (\vec{E} + \vec{E}^*); \quad \text{Re}\, \vec{H} = \tfrac{1}{2} (\vec{H} + \vec{H}^*) \\
\therefore \quad &\vec{S} = \text{Re}\, \vec{E} \times \text{Re}\, \vec{H} \\
&= \tfrac{1}{2} \left[\vec{E} \times \vec{H} + \vec{E}^* \times \vec{H}^* + \vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} \right]
\end{aligned} \tag{8.10.2}$$

Since $\int_0^T e^{\pm 2j\omega t} dt = 0$, the time-averages of the first two terms on r.h.s of (8.10.2) vanish over a complete period $T = 2\pi/\omega$.

$$\langle \vec{S} \rangle = \tfrac{1}{4} (\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H}) = \tfrac{1}{2} \, \text{Re} \, (\vec{E} \times \vec{H}^*) \tag{8.10.3}$$

• The magnitude of $\langle \vec{S} \rangle$, i.e., $|\langle \vec{S} \rangle|$ is called the ***intensity of radiation, I.*** Then, the ***radiation pressure*** p is given by

$$\boxed{p = \frac{I}{c} = \frac{|\langle \vec{S} \rangle|}{c}} \tag{8.10.4}$$

where c is the velocity of electromagnetic waves in vacuo.

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8.11 Characteristics of e.m. waves in vacuo

The eletromagnetic waves in free space show a number of characteristics, as observe by our analysis so far.

1. The e.m. waves are *tansverse in nature.* The electric field $\vec{E}$, the magnetic fie $\vec{H}$ and the direction of propagation of the wave are at right angles to each other.

2. The wave velocity c of e.m. waves in free space is given by $c = \sqrt{1/\varepsilon_0\mu_0}$, wher ε_0 = permittivity, and μ_0 = permeability of free space.

3. The rate of flow of electromagnetic energy per unit area in the direction of wav propagation is given by *Poynting vector,* $\vec{S} = \vec{E} \times \vec{H}$ or $(\vec{E} \times \vec{B})/\mu$.

4. The field vector $\vec{E}$ and $\vec{H}$ are *in phase* implying that when $\vec{E}$ attains th maximum value $\vec{E}_0$, $\vec{H}$ also attains its maximum value $\vec{H}_0$.

5. The electromagnetic energy is *equally shared* between the electric and th magnetic fields : $\frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\mu_0 H^2$ so that the *total energy density* U of e. m. waves is

$$U = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2}\mu_0 H^2 = \varepsilon_0 E^2.$$

- In dielectric, the *energy density* of e.m. waves is given by

$$U = U_e + U_m = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2 = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\varepsilon E^2 = \varepsilon E^2$$
$$= \varepsilon_r(\varepsilon_0 E^2) = \varepsilon_r \times \textit{energy density in free space.}$$

where ε_r = relative permittivity $= \varepsilon/\varepsilon_0$.

8.12 Characterization of light wave by electric vector

A light wave is an electromagnetic wave. The force exerted by its electric field vecto $\vec{E}$ on a particle of charge q is $q\vec{E}$, i.e., of magnitude qE. If however the particle moves with a velocity $\vec{v}$, the force exerted by the magnetic vector $\vec{B}$ of e.m. wave is $q(\vec{v} \times \vec{B})$ i.e., having a maximum value qvB.

Now, for an electromagnetic wave, we have

$$\left|\frac{\vec{E}}{\vec{H}}\right| = \sqrt{\frac{\mu}{\varepsilon}} = \mu\sqrt{\frac{1}{\mu\varepsilon}} = \mu c \quad \therefore \quad \frac{|\vec{E}|}{|\vec{B}|} = c \;\Rightarrow\; B = \frac{E}{c}$$

$$\therefore \quad qvB = \frac{v}{c}qE \qquad (8.12.1)$$

which shows that *magnetic force is* (v/c) *times the electric force.* Since $v \ll c$ this implies that electric force $\gg$ magnetc force. And that is why *a light wave is characterised by its electric vector* $\vec{E}$ which is also called the **light vector.** Every optical phenomenon—from the sensation of vision to photographic action — is due to the electric vector.

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• To form an idea about the relative values of electric and magnetic forces, let us consider the light intensity of sunlight on the earth's surface. It is about 1 kW/m^2, giving 1 kV/m as the amplitude of electric vector $\vec{E}$, which can be easily measured. The corresponding amplitude of magnetic vector $\vec{H}$ is about 3×10^{-6} T, too weak to be accurately measured.

8.13 Electromagnetic waves in a conducting medium

Consider a linear, homogeneous, isotropic conducting medium of conductivity σ, having no free charge.

For such a medium, the wave equation, as already obtained, are

$$\nabla^2\vec{E} - \sigma\mu\frac{\partial\vec{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0 \tag{8.13.1}$$

$$\nabla^2 H - \sigma\mu\frac{\partial\vec{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} = 0 \tag{8.13.2}$$

Both the equations contain damping terms proportional to σ, the conductivity of the medium.

Plane wave solutions — The above equations admit plane wave solutions. Assuming the field vectors to vary harmonically with time,

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)} \tag{8.13.3}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)} \tag{8.13.4}$$

where $\vec{E}_0$, $\vec{H}_0$ are complex wave amplitudes and $\vec{k} = k\hat{n}$, the propagation vector.

Substituting (8.13.3) in (8.13.1), we obtain

$$(-k^2 + \varepsilon\mu\omega^2 + j\sigma\mu\omega)\vec{E} = 0$$

$$\Rightarrow \quad k^2 = \varepsilon\mu\omega^2 + j\sigma\mu\omega \tag{8.13.5}$$

In *a conducting medium*, $\vec{k}$ is thus complex and let us take

$$k = \alpha + j\beta,$$

where the quantities α and β are both real.

$$\therefore \quad k^2 = \alpha^2 - \beta^2 + j2\alpha\beta \tag{8.13.6}$$

Comparing (8.13.5) and (8.13.6), we have

$$\alpha^2 - \beta^2 = \varepsilon\mu\omega^2,$$

$$\text{and} \qquad 2\alpha\beta = \sigma\mu\omega$$

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Solving these two simultaneous equations, we obtain

$$\alpha = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{\frac{1}{2}} \tag{8.13.7}$$

$$\beta = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{\frac{1}{2}} \tag{8.13.8}$$

We take the positive square root only, as negative sign would make α, β complex. In terms of α, β the field vectors becomes, using (8.13.3) and (8.13.4)

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{-\beta r}.\, e^{j(\alpha r-\omega t)} \tag{8.13.9}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{-\beta r} e^{j(\alpha r-\omega t)} \tag{8.13.10}$$

The equations show an important fact that *a plane wave cannot propagate in a conducting medium without attenuation of its amplitude* (Fig. 8.5). When a plane wave propagates in a conductor, the oscillating $\vec{E}$ in the wave sets up current in it; and to drive the current, work must be done and some energy is dissipated as heat (joule-loss). *This results in the attenuation* of the amplitude of the wave. The quantity β *is a measure of attenuation* and is called the **absorption coefficient** of the medium.

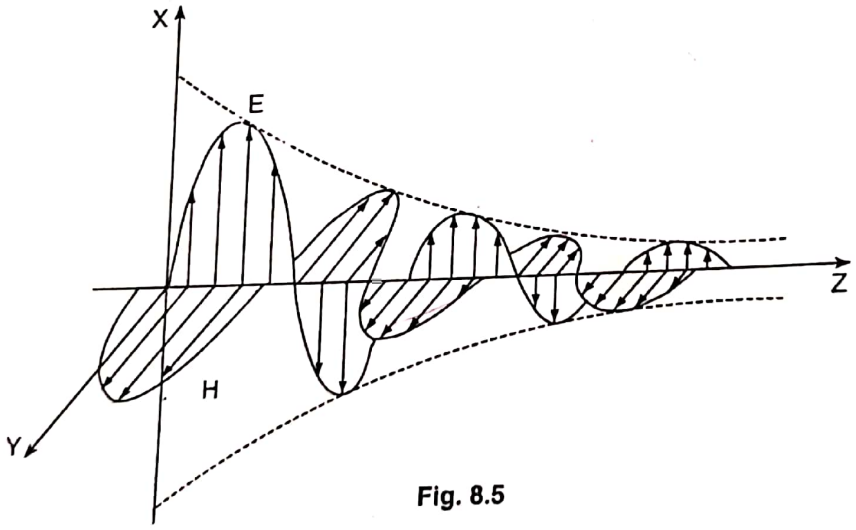

Fig. 8.5

• For a *non-conduting medium*, $\sigma = 0 \Rightarrow \beta = 0$ and there is *no attenuation*. For a **good conductor**, $\sigma/\varepsilon\omega \gg 1$ and we obtain

$$\boxed{\alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}}} \tag{8.13.11}$$

using the relation $2\alpha\beta = \sigma\mu\omega$.

• **Skin depth** — We have just seen the attenuation of amplitude of e.m. waves with distance inside a conductor.

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Skin depth δ is defined as that distance which the e.m. waves require in a conductor to reduce its amplitude by a factor $1/e$.

This occurs, from (8.13.9), when $r = 1/\beta = \sqrt{2/\mu\sigma\omega}$

$$\therefore \quad \textbf{Skin depth,} \quad \boxed{\delta = \sqrt{\frac{2}{\mu\sigma\omega}}} \tag{8.13.12}$$

Note that the skin depth is *inversely* proportional to the *square root of the frequency.* So, for waves of high frequency, δ is low. Thus, *the skin depth decreases as the frequency increases and vice versa.*

For instance, at microwave frequencies, δ for silver is extremely small $\sim 10^{-3}$ mm. The electric field of high frequency e.m. waves is thus confined to a small region on the surface of the conductor. This is why the phenomenon is called **skin effect**. Thus, *by using a thin coating of silver or copper, a poor conductor can be made a good conductor,* a technique used to reduce the material cost of good conductors.

• The skin depth of sea water at a frequency of 30 kHz is 10^{-1} m and explains why the *radio-communication with submarines becomes difficult at a depth of several metres.*

Substituting the solutions (8.13.3) and (8.13.4) in Maxwell's equations,

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{H} = 0, \\ \vec{k}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{k}\cdot H = 0. \end{aligned}\right\} \tag{8.13.13}$$

This implies that $\vec{E}$ and $\vec{H}$ *are normal to the direction of propagation.*

Substituting the solutions, again, in Maxwell's equations

$$\left.\begin{aligned} \vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \\ \text{and} \quad \vec{\nabla}\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \end{aligned}\right\}$$

$$\left.\begin{aligned} j\vec{k}\times\vec{E} = j\omega\mu\vec{H} \quad \Rightarrow \quad \vec{k}\times\vec{E} = \mu\omega\vec{H} \\ \text{and} \quad j\vec{k}\times\vec{H} = \sigma\vec{E} - j\omega\varepsilon\vec{E} \quad \Rightarrow \quad \vec{k}\times\vec{H} = -(\varepsilon\omega + j\sigma)\vec{E} \end{aligned}\right\} \tag{8.13.14}$$

implying that $\vec{E}$ and $\vec{H}$ *are mutually perpendicular, and they are also perpendicular to the direction of propagation.*

The attenuation of the amplitude for a progressive e.m. wave is shown graphically in Fig. 8.5.

Phase lag — From (8.13.14), we have

$$\vec{H} = \frac{1}{\mu\omega}(\alpha + j\beta)(\hat{n}\times\vec{E}) \quad (\because\ \vec{k} = k\hat{n};\ k = \alpha + j\beta)$$

$$= \frac{1}{\mu\omega}\sqrt{\alpha^2+\beta^2}\cdot e^{j\tan^{-1}(\beta/\alpha)}(\hat{n}\times\vec{E}). \tag{8.13.15}$$

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